arXiv:1210.6565v1 [math.AP] 24 Oct 2012

L^p-INTEGRABILITY OF THE GRADIENT OF SOLUTIONS TO QUASILINEAR SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

LUBOMIRA G. SOFTOVA

ABSTRACT. The Dirichlet problem for a class of quasilinear elliptic systems of equations with small-BMO coefficients in Reifenberg-flat domain Ω is considered. The lower order terms supposed to satisfy controlled growth conditions in $\mathbf{u}$ and $D\mathbf{u}$. It is obtained L^p-integrability with $p>2$ of $D\mathbf{u}$ where p depends explicitly on the data. An analogous result is obtained also for the Cauchy-Dirichlet problem for quasilinear parabolic systems.

1. INTRODUCTION

In the present work we study the integrability properties of the weak solutions of the following Dirichlet problem

$$\begin{cases} D_\alpha\Big(A_{ij}^{\alpha\beta}(x)D_\beta u^j(x)+a_i^\alpha(x,\mathbf{u})\Big)=b_i(x,\mathbf{u},D\mathbf{u}) & \text{a.a. } x\in\Omega\\ \mathbf{u}(x)=0 & \text{on } \partial\Omega\end{cases}\tag{1}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 2$ is a bounded *Reifenberg flat* domain (see Definition 2). The principal coefficients are discontinuous with "small" discontinuity expressed in terms of their *bounded mean oscillation* (BMO) in Ω (cf. [20]). The matrix $\mathbf{A}(x)=\{A_{ij}^{\alpha\beta}(x)\}_{i,j\leq N}^{\alpha,\beta\leq n}$ verifies

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j\geq\lambda|\xi|^2\quad\forall\ \xi\in\mathbb{M}^{N\times n},\quad \|\mathbf{A}\|_{\infty,\Omega}\leq M\tag{2}$$

with some positive constants λ and M. The non linear terms

$$\mathbf{a}(x,\mathbf{u})=\{a_i^\alpha(x,\mathbf{u})\}_{i\leq N}^{\alpha\leq n}\quad\text{and}\quad \mathbf{b}(x,\mathbf{u},\mathbf{z})=\{b_i(x,\mathbf{u},\mathbf{z})\}_{i\leq N}$$

supposed to be Carathéodory functions for $x\in\Omega$, $\mathbf{u}\in\mathbb{R}^N$, $\mathbf{z}\in\mathbb{M}^{N\times n}$ and satisfy *controlled growth conditions*. Namely, for $|\mathbf{u}|,|\mathbf{z}|\to\infty$ we have

$$a_i^\alpha(x,\mathbf{u})=\mathcal{O}(\varphi_1(x)+|\mathbf{u}|^{\frac{n}{n+2}})\quad\text{and}\quad b_i(x,\mathbf{u},\mathbf{z})=\mathcal{O}(\varphi_2(x)+|\mathbf{u}|^{\frac{n+2}{n-2}})\tag{3}$$

with $\varphi_1\in L^p(\Omega)$, $p>2$ and $\varphi_2\in L^q(\Omega)$, $q>\frac{2n}{n+2}$.

1991 *Mathematics Subject Classification.* Primary 35J57; Secondary 35K51; 35B40.

Key words and phrases. Elliptic and parabolic divergence form systems, controlled growth conditions, BMO, Dirichlet data, Reifenberg flat domain.

Our aim is to show that the problem (1) satisfies the Calderón–Zygmund property when Ω is (δ, R)-Reifenberg flat and the coefficients are (δ, R)-vanishing in Ω. Precisely, each bounded weak solution $\mathbf{u} \in W_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$ of (1) gains better regularity from the data φ_1 and φ_2 and belongs to $W_0^{1,\min\{p,q^*\}} \cap L^\infty(\Omega; \mathbb{R}^N)$ where q^* is the Sobolev conjugate of q (see (4)).

Similar result is obtained also for the Cauchy-Dirichlet problem for the parabolic quasilinear system

$$\begin{cases} u_t^i - D_\alpha(A_{ij}^{\alpha\beta}(x,t) D_\beta u^j + a_i^\alpha(x,t,\mathbf{u})) = b_i(x,t,\mathbf{u},D\mathbf{u})) & \text{a.a. } (x,t) \in Q \\ \mathbf{u}(x,t) = 0 & (x,t) \in \partial Q \end{cases}$$

in a cylinder $Q = \Omega \times (0,T)$ where Ω is (δ, R)-Reifenberg flat and $\partial Q = \Omega \cup \{\partial\Omega \times (0,T)\}$ is the parabolic boundary.

The problem of integrability and regularity of the solutions of linear and quasilinear elliptic/parabolic equations and systems is widely studied. Let us start with the classical results concerning equations/systems with smooth coefficients presented in the monographs [24, 25]. In the scalar case, $N = 1$, the notorious results of De Giorgi [10] and Nash [30] assert Hölder regularity of the solutions of linear divergence form equations with only L^∞ principal coefficients. One remarkable result that permits to obtain higher integrability of the weak solutions is due to Gehring [16]. He studied integrability properties of functions satisfying the reverse Hölder inequality. It was noticed that some power of the gradient of the weak solutions satisfies local reverse Hölder inequality. Modifying the Gehring lemma, Giaquinta and Modica [18] firstly obtain higher integrability of solutions of divergence form quasilinear elliptic equations. For the sake of completeness we give this result as it is presented in the monograph by Giaquinta [17, Theorem V.2.3].

Theorem 1. *Suppose that $g \in L^q(\Omega)$, $F \in L^{q+\delta}(\Omega)$, $g, F \geq 0$, $q > 1$, $\delta > 0$ and*

$$\fint_{\mathcal{B}_R(x)} g^q dx \leq B \left(\fint_{\mathcal{B}_{2R}(x)} g dx \right)^q + \fint_{\mathcal{B}_{2R}(x)} F^q dx + \theta \fint_{\mathcal{B}_{2R}(x)} g^q dx$$

for a.a. $x \in \Omega$, $R < \frac{1}{2}\min\{d(x, \partial\Omega), R_0\}$ where $R_0 > 0$, $B > 1$, $\theta \in [0,1)$. Then $g \in L^{p,\mathrm{loc}}(\Omega)$ and

$$\left(\fint_{\mathcal{B}_R(x)} g^p dx \right)^{1/p} \leq C \left\{ \left(\fint_{\mathcal{B}_{2R}(x)} g^q dx \right)^{1/q} + \left(\fint_{\mathcal{B}_{2R}(x)} F^p dx \right)^{1/p} \right\}$$

for any ball $\mathcal{B}_{2R} \subset \Omega$, $2R < R_0$, $p \in [q, p_0)$ where $C > 0$, $p_0 > q$ depend only on B, θ, q, n.

There are various generalizations of the above theorem permitting to study elliptic and parabolic problems with Dirichlet and Neumann boundary conditions (see [2, 3, 4, 5, 11, 15, 27]). Other results concerning higher integrability of divergence form quasilinear equations and variational equations could be found in [9, 17, 19, 21, 28]. The L^p-estimates of derivatives obtained such way laid the foundation to the so-called "direct method" of proving partial regularity of solutions. Recently, the method of A-harmonic approximation permits to study the regularity of the solutions without the use of the Gehring lemma. For more details we refer the reader to [1, 12, 13, 14], see also the references therein.

The regularity theory for linear operators with smooth data was extended on operators with discontinuous coefficients defined in rough domains. In [6, 7, 8] the authors consider divergence form elliptic and parabolic equations and systems with BMO coefficients in Reifenberg flat domain with Dirichlet boundary conditions extending such way the known results on operators with VMO coefficients too (see also [28, 34, 35, 22, 23] and the references therein). In [15] a reverse Hölder inequality is established for quasilinear elliptic systems with principal coefficient being VMO in x and under controlled growth conditions over the lower order terms. It permits the authors to obtain interior Hölder continuity of solutions to scalar equations as well as partial Hölder regularity of solutions to systems. In [31, 32] global Hölder regularity of solutions to elliptic quasilinear equations with VMO in x principal coefficients is proved under *strictly controlled growth* conditions. Later this result is extended for quasilinear elliptic and parabolic equations in Reifenberg flat domains supposing *controlled growth* conditions and Dirichlet boundary data (see [11, 33, 36, 37]).

In the present work we extend the results from [37] to elliptic and parabolic systems with discontinuous data. Making use of the linear L^p-theory for systems, developed in [7, 8] and the bootstrap method we prove $D\mathbf{u} \in L^r$ with r depending explicitly on the data φ_1 and φ_2 in (3).

2. Elliptic systems, definitions and main result

In the following we use the standard notations:

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\rho > 0$ and $\mathcal{B}_\rho(x) = \{y \in \mathbb{R}^n : |x - y| < \rho\}$.
- let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $x \in \Omega$ and denote $\Omega_\rho(x) = \Omega \cap \mathcal{B}_\rho(x)$.
- $\mathbb{M}^{N\times n}$ is the set of $N \times n$-matrices.
- For a vector function $\mathbf{u} = (u^1, \dots, u^N) : \Omega \to \mathbb{R}^N$ we write

$$|\mathbf{u}|^2 = \sum_{j\le N} |u^j|^2, \quad D_\alpha u^j = \frac{\partial}{\partial x_\alpha} u^j,$$

$$D\mathbf{u} = \{D_\alpha u^j\}_{j\le N}^{\alpha\le n} \in \mathbb{M}^{N\times n}, \quad |D\mathbf{u}|^2 = \sum_{\substack{\alpha\le n\\ j\le N}} |D_\alpha u^j|^2.$$

- Let $f:\Omega\to\mathbb{R}$ and $|\Omega|$ be the Lebesgue measure of Ω, then

$$\fint_\Omega f(y)dy = \frac{1}{|\Omega|}\int_\Omega f(y)dy, \quad \|f\|_{p,\Omega}^p = \|f\|_{L^p(\Omega)}^p = \int_\Omega |f(y)|^p dy.$$

- For $\mathbf{u}\in L^p(\Omega;\mathbb{R}^N)$ write $\|\mathbf{u}\|_{p,\Omega}$ instead of $\|\mathbf{u}\|_{L^p(\Omega;\mathbb{R}^N)}$.
- For each $s\in(1,\infty)$ recall that s^* means the Sobolev conjugate of s

$$s^* = \begin{cases} \frac{ns}{n-s} & \text{if } s<n \\ \text{arbitrary large number } >1 & \text{if } s\ge n. \end{cases} \tag{4}$$

For the function spaces we follow the notions of the monographs [24, 28]. Through all the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various constants and may change from one occurrence to another.

In [38] Reifenberg introduced a class of domains with rough boundary that can be approximated by hyperplanes at every point and at every scale. Namely

Definition 2. The domain Ω is (δ,R)-Reifenberg flat if there exist positive constants R, $\delta<1$ such that for each $x\in\partial\Omega$ and each $\rho\in(0,R)$ there is a local coordinate system $\{y_1,\dots,y_n\}$ with the property

$$\mathcal{B}_\rho(x)\cap\{y_n>\delta\rho\}\subset\Omega_\rho(x)\subset\mathcal{B}_\rho(x)\cap\{y_n>-\delta\rho\}. \tag{5}$$

Reifenberg arrived at that concept of flatness in his studies on Plateau's problem in higher dimensions and he proved that such a domain is locally a topological disc when δ is small enough. It is easy to see that a C^1-domain is a Reifenberg flat with $\delta\to 0$ as $R\to 0$. A domain with Lipschitz boundary with a Lipschitz constant less than δ also verifies the condition (5) if δ is small enough (say $\delta<1/8$). But the class of Reifenberg's domains is much more wider and contain domains with fractal boundaries. For instance, consider a self-similar snowflake S_β. It is a flat version of the Koch snowflake $S_{\pi/3}$ where the angle of the spike with respect to the horizontal is β. A domain $\Omega\subset\mathbb{R}^2$ with $S_\beta\subset\partial\Omega$ is a Reifenberg flat if $0<\sin\beta<\delta<1/8$. This kind of flatness exhibits minimal geometrical conditions necessary for some natural properties in analysis and potential theory to hold. For more detailed overview of the properties of these domains we refer the reader to the papers [29, 39].

From (5) it follows that $\partial\Omega$ satisfies the (A)-property (cf. [9, 17, 24]). Precisely, the measure $|\Omega_\rho(x)|$ is δ-comparable to $|\mathcal{B}_\rho(x)|$, that is there exists a positive constant

$A(\delta) < 1/2$ such that

$$A(\delta)|\mathcal{B}_\rho(x)| \leq |\Omega_\rho(x)| \leq (1 - A(\delta))|\mathcal{B}_\rho(x)| \tag{A}$$

for any fixed $x \in \partial\Omega$, $\rho \in (0, R)$ and $\delta \in (0, 1)$. This condition excludes that Ω may have sharp outward and inward cusps. Moreover, for small δ they can be approximated in a uniform way by Lipschitz domains with a Lipschitz constant less then δ (see [8, Lemma 5.1]). As consequence, they are $W^{1,p}$-extension domains, $1 \leq p \leq \infty$, hence the usual extension theorems, the Sobolev and Sobolev–Poincaré inequalities are valid in Ω.

To describe the discontinuity of the principal coefficients we need of the following

Definition 3. *We say that a function $a(x)$ is a (δ, R)-vanishing if there exist positive constants R and $\delta < 1$ such that*

$$\sup_{0<\rho\leq R}\sup_{x\in\Omega} ⨍_{\Omega_\rho(x)} |a(y) - \overline{a}_{\Omega_\rho(x)}|^2 dy \leq \delta^2, \quad \overline{a}_{\Omega_\rho(x)} = ⨍_{\Omega_\rho(x)} a(y)dy. \tag{6}$$

We suppose that all $A_{ij}^{\alpha\beta}(x)$ are (δ, R)-vanishing. It implies that $\mathbf{A} \in BMO(\Omega)$ with a small BMO norm $\|\mathbf{A}\|_* < \delta$.

The nonlinear terms $\mathbf{a}(x, \mathbf{u})$ and $\mathbf{b}(x, \mathbf{u}, \mathbf{z})$ are Carathéodory functions for $x \in \Omega$, $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{z} \in \mathbb{M}^{N\times n}$ and satisfy the *controlled growth conditions*

$$|\mathbf{a}(x, \mathbf{u})| \leq \Lambda(\varphi_1(x) + |\mathbf{u}|^{\frac{n}{n-2}}), \quad \varphi_1 \in L^p(\Omega),\ p > 2 \tag{7}$$

$$|\mathbf{b}(x, \mathbf{u}, \mathbf{z})| \leq \Lambda\Big(\varphi_2(x) + |\mathbf{u}|^{\frac{n+2}{n-2}} + |\mathbf{z}|^{\frac{n+2}{n}}\Big), \quad \varphi_2 \in L^q(\Omega),\ q > \frac{2n}{n+2} \tag{8}$$

with some positive constant Λ. In the particular case $n = 2$ the powers of $|\mathbf{u}|$ could be arbitrary positive numbers while the growth of $|\mathbf{z}|$ is quadratic (cf. [17, 24]).

Under a *weak solution* to the problem (1) we mean a function $\mathbf{u} \in W_0^{1,p}(\Omega; \mathbb{R}^N)$, $1 < p < \infty$ satisfying

$$\begin{aligned}\int_\Omega A_{ij}^{\alpha\beta}(x) D_\beta u^j(x) D_\alpha \chi^i(x)dx &+ \sum_{\substack{\alpha\leq n\\ i\leq N}} \int_\Omega a_i^\alpha(x, \mathbf{u}(x)) D_\alpha \chi^i(x)dx \\ &+ \int_\Omega b_i(x, \mathbf{u}(x), D\mathbf{u}(x))\chi^i(x)dx = 0, \quad j = 1, \dots, N\end{aligned}$$

for all $\chi \in W_0^{1,p'}(\Omega; \mathbb{R}^N)$, $p' = p/(p-1)$. The conditions (7) and (8) are the natural ones that ensure convergence of the integrals above. Moreover, they are optimal since a growth of the gradient greater than $\frac{n+2}{n}$ leads to unbounded solutions as it is seen from the following example (cf. [25, 32]). The function $u(x) \in W^{1,2}(\mathcal{B}_1(0))$, $u(x) = |x|^{\frac{r-2}{r-1}}$ is a solution of the equation $\Delta u = C|Du|^r$ in $\mathcal{B}_1(0)$. Note that $u(x) \notin L^\infty(\mathcal{B}_1(0))$ for $\frac{n+2}{n} < r < 2$.

In generally we cannot expect boundedness of each solution of (1) unless we add some structural conditions. Consider, for instance, the system

$$D_\alpha(A_i^\alpha(x,\mathbf{u},D\mathbf{u})) = b_i(x,\mathbf{u},D\mathbf{u}) \qquad x\in\Omega$$

where

$$A_i^\alpha(x,\mathbf{u},D\mathbf{u}) = \sum_{\beta\le n}\sum_{j\le N}(A_{ij}^{\alpha\beta}(x)D_\beta u^j + a_i^\alpha(x,\mathbf{u}))$$

are measurable in $x\in\Omega$. Assume a pointwise coercive and sign conditions, both of them for large values of the corresponding component of $\mathbf{u}$, precisely: for every $i\in\{1,\dots,N\}$ there exist constants $\theta^i, M^i, \nu\in(0,+\infty)$ such that for $u^i\ge\theta^i$ we have

$$\begin{cases} \nu|\xi^i|^2 - M^i \le \displaystyle\sum_{\alpha\le n} A_i^\alpha(x,\mathbf{u},\xi)\xi_i^\alpha \\ 0\le b_i(x,\mathbf{u},\xi) \quad \text{for a.a. } x\in\Omega,\ \forall\xi\in\mathbb{M}^{N\times n}. \end{cases} \tag{9}$$

Suppose (7), (8) and (9) and let $\mathbf{u}\in W^{1,2}\cap L^{\frac{2n}{n-2}}(\Omega;\mathbb{R}^N)$ be a weak solution of (1) then for each $i\in\{1,\dots,N\}$

$$\sup_\Omega u^i \le \theta^i + K^i$$

where K^i depend on $M^i, n, |\Omega|$ and ν (see [26]).

Theorem 4. *Let $\mathbf{u}\in W_0^{1,2}\cap L^\infty(\Omega;\mathbb{R}^N)$ be a weak solution of the problem* (1) *under the conditions* (2)*,* (7) *and* (8)*. Then there exists a small number $\delta_0>0$ such that if Ω is (δ,R)-Reifenberg flat domain and $A_{ij}^{\alpha\beta}(x)$ are (δ,R)-vanishing with $\delta<\delta_0<1$ then*

$$\mathbf{u}\in W_0^{1,r}\cap L^\infty(\Omega;\mathbb{R}^N) \quad \textit{with} \quad r=\min\{p,q^*\}. \tag{10}$$

Proof. In [17, Chapter 5] Giaquinta considers quasilinear strongly elliptic systems with L^∞ principal coefficients, under the conditions (7) and (8). Making use of the reverse Hölder's inequality and the version of the Gehring lemma it is shown that there exists an exponent $r_0>2$ such that $\mathbf{u}\in W^{1,r_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ (cf. [17, Theorem V.2.3],[9, Chapter III] or [28, Lemma 3.2.23]). Since, roughly speaking, Caccioppoli-type inequalities hold up to the boundary, the method for obtaining higher integrability can be carried over up to the boundary. In [17, Chapter 5] it is done for the Dirichlet problem in Lipschitz domain. Since the Reifenberg flat domain can be uniformly approximated by Lipschitz domains the same result still holds true. Precisely, there is $r_0>2$ such that

$$\|D\mathbf{u}\|_{r,\Omega}\le N \qquad \forall\ r\in[2,r_0) \tag{11}$$

where N and r_0 depend on $n,\Lambda,\lambda,\|\varphi_1\|_{p,\Omega},\|\varphi_2\|_{q,\Omega},|\Omega|,\|D\mathbf{u}\|_{2,\Omega}$.

Let $n>2$ and $\mathbf{u}\in W_0^{1,2}\cap L^\infty(\Omega;\mathbb{R}^N)$ be a solution of (1). Fixing that solution in the nonlinear terms we get the linearized problem

$$
\begin{cases} D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j)=f_i(x)-\operatorname{div}(\mathbb{A}_i(x)) & \text{a.a. } x\in\Omega\\ \mathbf{u}(x)=0 & \text{on } \partial\Omega \end{cases} \tag{12}
$$

where

$$
\begin{aligned} f_i(x)=&b_i(x,\mathbf{u},D\mathbf{u}), \qquad \mathbf{f}(x)=\mathbf{b}(x,\mathbf{u},D\mathbf{u}),\\ \mathbb{A}_i(x)=&(a_i^1(x,\mathbf{u}),\dots,a_i^n(x,\mathbf{u})), \qquad \mathbb{A}(x)=(\mathbb{A}_1(x),\dots,\mathbb{A}_N(x)) \end{aligned}
$$

and by (7), (8) and (11) we get

$$
\begin{cases} \|\mathbb{A}\|_{p,\Omega}\le C\left(\|\varphi_1\|_{p,\Omega}+\|\mathbf{u}\|_{\infty,\Omega}^{\frac{n}{n+2}}\right)\\ \|\mathbf{f}\|_{q_1,\Omega}\le C\left(\|\varphi_2\|_{q_1,\Omega}+\|\mathbf{u}\|_{\infty,\Omega}^{\frac{n+2}{n-2}}+\|D\mathbf{u}\|_{\frac{q_1(n+2)}{n},\Omega}^{\frac{n+2}{n}}\right) \end{cases} \tag{13}
$$

with $p>2$ and $q_1=\min\left\{q,\frac{r_0n}{n+2}\right\}$. Further, for all $f_i\in L^{q_1}(\Omega)$, $i=1,\dots,N$ there exists a vector field $\mathbb{F}_i(x)\in L^{q_1^*}(\Omega,\mathbb{R}^n)$ such that $f_i(x)=\operatorname{div}\mathbb{F}_i(x)$. Denote $\mathbb{F}(x)=(\mathbb{F}_1(x),\dots,\mathbb{F}_N(x))$, then by [32, Lemma 3.1]) we have

$$
\|\mathbb{F}\|_{q_1^*,\Omega}\le C\|\mathbf{f}\|_{q_1,\Omega}, \quad q_1=\min\left\{q,\frac{r_0n}{n+2}\right\}. \tag{14}
$$

Thus the problem (12) becomes

$$
\begin{cases} D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j(x))=\operatorname{div}(\mathbb{F}_i(x)-\mathbb{A}_i(x)) & \text{a.a. } x\in\Omega\\ \mathbf{u}(x)=0, & x\in\partial\Omega. \end{cases} \tag{15}
$$

For linear systems as above we dispose with the regularity result of Byun and Wang [8, Theorem 1.7] that asserts there exists a small positive constant $\delta=\delta(\lambda,p,n,N)$ such that for each (δ,R)-vanishing $A_{ij}^{\alpha\beta}$, for each (δ,R)-Reifenberg flat Ω, and for each matrix function $\mathbb{F}-\mathbb{A}\in L^{r_1}(\Omega;\mathbb{M}^{N\times n})$, with $r_1=\min\{p,q_1^*\}$, the solution $\mathbf{u}\in W_0^{1,2}\cap L^\infty(\Omega;\mathbb{R}^N)$ of (15) belongs to $W_0^{1,r_1}\cap L^\infty(\Omega;\mathbb{R}^N)$ and the following estimate holds

$$
\|D\mathbf{u}\|_{r_1,\Omega}\le C\|\mathbb{F}-\mathbb{A}\|_{r_1,\Omega}, \qquad r_1=\min\{p,q_1^*\} \tag{16}
$$

with $C=C(\lambda,p,n,N,|\Omega|)$.

Our goal is to show the inclusion $D\mathbf{u}\in L^r(\Omega;\mathbb{M}^{N\times n})$ with $r=\min\{p,q^*\}$. For this we study the following cases:

1) If $q\le\frac{r_0n}{n+2}$ then $q_1=q$ in (14) and $r_1\equiv r=\min\{p,q^*\}$.

2) If $q > \frac{r_0 n}{n+2}$, then $q_1 = \frac{r_0 n}{n+2}$ and

$$q_1^* = \begin{cases} \dfrac{r_0 n}{n+2-r_0} & \text{if } \dfrac{r_0 n}{n+2} < n \\ \text{arbitrary large number } > 1 & \text{if } \dfrac{r_0 n}{n+2} \geq n. \end{cases}$$

Consider again two sub-cases:

2_a) If $n > \frac{r_0 n}{n+2}$ then $r_1 = \min\{p, \frac{r_0 n}{n+2-r_0}\}$. If $r_1 = p$ then the theorem holds true otherwise $D\mathbf{u} \in L^{r_1}(\Omega : \mathbb{M}^{n\times N})$ with $r_1 = \frac{r_0 n}{n+2-r_0}$.

2_b) If $n \leq \frac{r_0 n}{n+2}$ then q_1^* is arbitrary large number, that implies $r_1 = p$ and the theorem holds true once again.

It is easy to see that $r_1 \equiv r$ unless

$$\frac{r_0 n}{n+2} < q \qquad \text{and} \qquad \frac{r_0 n}{n+2} < n$$

when $\mathbf{u} \in W_0^{1,r_1} \cap L^\infty(\Omega; \mathbb{R}^N)$ with $r_1 = \frac{r_0 n}{n+2-r_0}$. It holds for any solution of the linearized problem (15) including the one fixed in the coefficients in (1).

Consider once again (13) with $D\mathbf{u} \in L^{r_1}(\Omega; \mathbb{M}^{N\times n})$. Hence $f_i \in L^{q_2}(\Omega)$ with $q_2 = \min\{q, \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)}\}$ and the associated vector-field $\mathbb{F}_i$ belongs to $L^{q_2^*}(\Omega; \mathbb{R}^n)$. Than $\mathbb{F}_i - \mathbb{A}_i \in L^{r_2}(\Omega; \mathbb{R}^n)$ with $r_2 = \min\{p, q_2^*\}$. Applying [8, Theorem 1.7] to system (15) and repeating the same procedure as above we get that the theorem holds with $r_2 \equiv r$ if

(17)
$$i) \ \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} \geq n \qquad \text{or} \qquad ii) \ \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} \geq q.$$

Otherwise $r_2 = \frac{r_0 n^2}{(n+2)^2 - r_0(n+2) - r_0 n}$ if

$$\frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < q \qquad \text{and} \qquad \frac{r_0 n^2}{(n+2)^2 - r_0(n+2)} < n \tag{18}$$

Repeating the same procedure k-times we get that the assertion holds if

$$\frac{r_0 n^k}{(n+2)^k - r_0 \sum_{s=0}^{k-2} n^s (n+2)^{k-1-s}} \geq \min\{n, q\}\,. \tag{19}$$

Direct calculations give that (19) is equivalent to

$$k > \min\left\{ \Big[\log \frac{r_0}{r_0 - 2} \Big/ \log \frac{n+2}{n} \Big], \Big[\log \frac{r_0(2q + qn + 2)}{q(n+2)(r_0 - 2)} \Big/ \log \frac{n+2}{n} \Big] + 1 \right\}$$

where $[x]$ means the integer part of x.

The case $n = 2$ is simpler and is left to the reader. $\square$

3. Quasilinear parabolic systems

Let $Q = \Omega \times (0,T)$ be a cylinder in $\mathbb{R}^{n+1}$ with Ω being (δ, R)-Reifenberg flat. Denote by $\mathcal{C}_\rho$ the parabolic cylinder

$$\mathcal{C}_\rho(x,t) = \mathcal{B}_\rho(x) \times (t-\rho^2, t), \quad Q_\rho(x,t) = Q \cap \mathcal{C}_\rho(x,t) \quad \text{ for } (x,t) \in Q,$$

$$\overline{a}_{Q_\rho(x,t)} = \fint_{Q_\rho(x,t)} a(y,\tau)dyd\tau = \frac{1}{|Q_\rho(x,t)|}\int_{Q_\rho(x,t)} a(y,\tau)dyd\tau.$$

Let $1 < r < \infty$ and $\mathbf{u} : Q \to \mathbb{R}^N$.

1. The space $W_r^{1,0}(Q;\mathbb{R}^N)$ consists of all functions $\mathbf{u} \in L^r(Q;\mathbb{R}^N)$ having a finite norm

$$\|\mathbf{u}\|^r_{W_r^{1,0}(Q;\mathbb{R}^N)} = \|\mathbf{u}\|^r_{r,Q} + \|D\mathbf{u}\|^r_{r,Q}\,.$$

2. The space $W_*^{1,r}(Q;\mathbb{R}^N) = L^r(0,T;W^{1,r}(\Omega;\mathbb{R}^N)) \cap W^{1,r}(0,T;W^{-1,r'}(\Omega;\mathbb{R}^N))$, $r' = r/(r-1)$ consists of the functions $\mathbf{u} \in W_r^{1,0}(Q;\mathbb{R}^N)$ for which there exist vector functions $\mathbf{g} \in L^r(Q;\mathbb{R}^N)$ and $\mathbf{F} \in L^r(Q;\mathbb{M}^{N\times n})$ such that

$$\mathbf{u}_t = \mathrm{div}\mathbf{F} - \mathbf{g} \qquad \text{a.e. in } Q$$

in the sense of distributions, that is, for each vector function $\chi \in C_0^\infty(Q)$ with $\chi(x,T) = 0$ holds

$$\int_Q \mathbf{u}\cdot\chi_t\,dxdt = \int_Q (\mathbf{F}\cdot D\chi + \mathbf{g}\cdot\chi)\,dxdt\,. \tag{20}$$

The space $W_*^{1,r}(Q;\mathbb{R}^N)$ is endowed by the norm

$$\|\mathbf{u}\|_{W_*^{1,r}(Q)} = \|\mathbf{u}\|_{W_r^{1,0}(Q)} + \inf\left\{\left(\int_Q |\mathbf{F}|^r + |\mathbf{g}|^r\right)^{1/r}\right\}$$

where the infimum is taken over all $\mathbf{F}$ and $\mathbf{g}$ satisfying (20). The closure of $C_0^\infty(Q)$ with respect to this norm is denoted by $\overset{\circ}{W}{}_*^{1,r}(Q;\mathbb{R}^N)$.

3. $V_2(Q;\mathbb{R}^N)$ stands for the Banach space of all functions $\mathbf{u} \in W_2^{1,0}(Q;\mathbb{R}^N)$ for which

$$\|\mathbf{u}\|_{V_2(Q;\mathbb{R}^N)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|\mathbf{u}(\cdot,t)\|_{2,\Omega} + \|D\mathbf{u}\|_{2,Q} < \infty\,.$$

4. $V_2^{1,0}(Q;\mathbb{R}^N)$ consists of all $\mathbf{u} \in V_2(Q;\mathbb{R}^N)$ that are continuous in t with respect to the norm of $L^2(\Omega;\mathbb{R}^N)$

$$\lim_{\Delta t\to 0} \|\mathbf{u}(\cdot,t+\Delta t) - \mathbf{u}(\cdot,t)\|_{2,\Omega} = 0.$$

The norm in $V_2^{1,0}(Q;\mathbb{R}^N)$ is given by

$$\|\mathbf{u}\|_{V_2^{1,0}(Q;\mathbb{R}^N)} = \max_{t\in[0,T]} \|\mathbf{u}(\cdot,t)\|_{2,\Omega} + \|D\mathbf{u}\|_{2,Q}.$$

We consider the Cauchy-Dirichlet problem for the strongly parabolic quasilinear system

$$
\begin{cases} u^i_t - D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j + a^\alpha_i(x,t,\mathbf{u})) = b_i(x,t,\mathbf{u},D\mathbf{u}) & \text{a.a. } (x,t)\in Q \\ \mathbf{u}(x,t)=0 & (x,t)\in\partial Q. \end{cases} \tag{21}
$$

The principal coefficients satisfy $A^{\alpha\beta}_{ij}\in L^\infty(Q)$ and

$$
A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \ge \nu|\xi|^2 \qquad \forall\, \xi\in\mathbb{M}^{N\times n}, \quad \nu=\text{const}>0. \tag{22}
$$

In addition we suppose that $A^{\alpha\beta}_{ij}$ are (δ,R)-vanishing, that is

$$
\sup_{0<\rho\le R}\sup_{(x,t)\in Q} \fint_{Q_\rho(x,t)} |A^{\alpha\beta}_{ij}(y,\tau) - \overline{A^{\alpha\beta}_{ij}}_{Q_\rho(x,t)}|^2 dy d\tau \le \delta^2 \tag{23}
$$

which implies small BMO norm of each $A^{\alpha\beta}_{ij}$.

The functions $a^\alpha_i(x,t,\mathbf{u})$, $b_i(x,t,\mathbf{u},\mathbf{z})$ are Carathéodory ones and verify the *controlled growth conditions* (see [2, 25])

$$
\sum_{\substack{\alpha\le n\\ i\le N}} |a^\alpha_i(x,t,\mathbf{u})| \le \Lambda(\psi_1(x,t)+|\mathbf{u}|^{\frac{n+2}{n}}) \tag{24}
$$

$$
\sum_{i\le N} |b_i(x,t,\mathbf{u},\mathbf{z})| \le \Lambda(\psi_2(x,t)+|\mathbf{u}|^{\frac{n+4}{n}}+|\mathbf{z}|^{\frac{n+4}{n+2}})\,. \tag{25}
$$

with $\psi_1\in L^p(Q), p>2$ and $\psi_2\in L^q(Q), q>\frac{2(n+2)}{n+4}$.

A vector function $\mathbf{u}\in \overset{\circ}{W}{}^{1,2}_{*}\cap L^\infty(Q;\mathbb{R}^N)$ is a weak solution to (21) if for any function $\chi\in \overset{\circ}{W}{}^{1,2}_{*}(Q;\mathbb{R}^N)$, $\chi(x,T)=0$ we have

$$
\int_Q u^i(x,t)\chi_t(x,t)dxdt - \int_Q (A^{\alpha\beta}_{ij}(x,t)D_\beta u^j(x,t) + a^\alpha_i(x,t,\mathbf{u}))D_\alpha\chi^i(x,t)dxdt + \int_Q b_i(x,t,\mathbf{u},D\mathbf{u})\chi^i(x,t)dxdt = 0\,.
$$

Theorem 5. *Let $\mathbf{u}\in \overset{\circ}{W}{}^{1,2}_{*}\cap L^\infty(Q;\mathbb{R}^N)$ be a weak solution to (21) under the conditions (22)-(25). Then there exists a small positive constant $\delta_0<1$ such that if Ω is (δ,R)-Reifenberg flat and $A^{\alpha\beta}_{ij}$ are (δ,R)-vanishing with $0<\delta<\delta_0$ then*

$$
\mathbf{u}\in \overset{\circ}{W}{}^{1,r}_{*}\cap L^\infty(Q;\mathbb{R}^N) \qquad \textit{with } r=\min\{p,q^{**}\}
$$

$$
\textit{where} \quad q^{**} = \begin{cases} \frac{q(n+2)}{n+2-q} & \textit{if } q<n+2 \\ \textit{arbitrary large number } >1 & \textit{if } q\ge n+2. \end{cases}
$$

Proof. The higher integrability of the gradient of the solution follows by the modification of the Gehring lemma due to Arkhipova [4, Theorem 1] which is very efficient for the study of parabolic systems with controlled growth conditions in domains with boundary $\partial\Omega$ satisfying a kind of (A)-property. Recently, similar result is obtained in [11] for domains having strongly Lipschitz boundary. Since the Reifenberg flat domain can be approximated uniformly with Lipschitz domains with a small Lipschitz constant [8, Lemma 5.1]) we have that there exists $r_0 > 2$ such that for any solution $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}_{*}(Q;\mathbb{R}^N)$ holds

$$\|D\mathbf{u}\|_{r,Q} \leq N \qquad \forall r \in [2, r_0)$$

where r_0 and N depend on the data of the problem and $\|D\mathbf{u}\|_{2,Q}$. Take a solution $\mathbf{u} \in \overset{\circ}{W}{}^{1,2}_{*} \cap L^\infty(Q;\mathbb{R}^N)$ of (21) and fix it in the lower order terms. Thus we get the linearized problem

$$\begin{cases} u^i_t - D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = f_i(x,t) + \operatorname{div} \mathbb{A}_i(x,t) & \text{a.a. } (x,t) \in Q \\ \mathbf{u}(x,t) = 0 & \text{on } \partial Q \end{cases} \tag{26}$$

where

$$\begin{aligned} &\mathbb{A}_i(x,t) = (a^1_i(x,t,\mathbf{u}),\dots,a^n_i(x,t,\mathbf{u})), \quad \mathbf{A}(x,t) = \{a^\alpha_i(x,t,\mathbf{u}(x,t))\}^{\alpha\leq n}_{i\leq N} \\ &f_i(x,t) = b_i(x,t,\mathbf{u},D\mathbf{u}), \quad \mathbf{f} = (f_1(x,t),\dots,f_N(x,t)). \end{aligned}$$

Making use of the conditions (24) and (25) we get

$$\begin{aligned} &\sum_{i\leq N} \|\mathbb{A}_i\|_{p,Q} \leq C\Big(\|\psi_1\|_{p,Q} + \|\mathbf{u}\|^{\frac{n+2}{n}}_{\infty,Q}\Big) \\ &\sum_{i\leq N} \|f_i\|_{q_1,Q} \leq C\Big(\|\psi_2\|_{q_1,Q} + \|\mathbf{u}\|^{\frac{n+4}{n}}_{\infty,Q} + \|D\mathbf{u}\|^{\frac{n+4}{n+2}}_{\frac{q_1(n+4)}{n+2}}\Big) \end{aligned}$$

with $p > 2$, $q_1 = \min\Big\{q, \frac{r_0(n+2)}{n+4}\Big\}$. Let $\Omega' \subset \mathbb{R}^n$ be C^2-domain such that $\Omega \Subset \Omega'$ and consider the cylinder $Q' = \Omega' \times (0,T)$. Suppose that $f_i(x,t)$ is extended as zero out of Q. It is well known (cf. [25]) that for each $f_i \in L^{q_1}(Q')$ the linear problem

$$\begin{cases} F^i_t - \Delta F^i = F^i_t - D_\alpha(\delta^{\alpha\beta} D_\beta F^i) = f_i(x,t) & \text{a.a. } (x,t) \in Q' \\ F^i(x,t) = 0 & \text{on } \partial Q' \end{cases}$$

has a unique solution $F^i \in \overset{\circ}{W}{}^{2,1}_{q_1}(Q')$ and the estimate holds

$$\|F^i\|_{q_1,Q} \leq \|F^i\|_{\overset{\circ}{W}{}^{2,1}_{q_1}(Q')} \leq C\|f_i\|_{q_1,Q}.$$

Denote by $\mathbf{F} = (F^1,\dots,F^N)$, $\mathbb{F}^\alpha_i = (A^{\alpha\beta}_{ij} - \delta^{\alpha\beta}\delta_{ij})D_\beta F^j$ and $\mathbb{F}_i = (\mathbb{F}^1_i,\dots,\mathbb{F}^n_i)$. Since the Sobolev trace theorem holds for domain with Reifenberg flat boundary we

get that $F^i|_S \in W^{2-\frac{1}{q_1},1-\frac{1}{2q_1}}(S)$ where $S = \partial\Omega \times (0,T)$. Consider the linear system

$$(27)\qquad \begin{cases} (u^i - F^i)_t - D_\alpha(A^{\alpha\beta}_{ij} D_\beta(u^j - F^j)) & \\ \qquad = D_\alpha((A^{\alpha\beta}_{ij} - \delta^{\alpha\beta}\delta_{ij})D_\beta F^j) + \operatorname{div}(\mathbb{A}_i) & \\ \qquad = \operatorname{div}(\mathbb{F}_i + \mathbb{A}_i) & \text{a.a. } (x,t) \in Q \\ \mathbf{u} - \mathbf{F} = -\mathbf{F} & \text{on } \partial Q. \end{cases}$$

By the imbedding theorems, $DF^i \in L^{q_1^{**}}$ (cf. [25, Ch.II,Lemma 3.3]) and

$$\|DF^i\|_{q_1^{**},Q} \le C\|f^i\|_{q_1,Q}$$

Hence $\mathbb{F}_i + \mathbb{A}_i \in L^{r_1}(Q)$ with $r_1 = \min\{p, q_1^{**}\}$. Applying [7, Corollary 2.10] on the linear problem (27) we get

$$\|D(\mathbf{u}-\mathbf{F})\|_{r_1,Q} \le C\Big(1 + \|D\mathbf{F}\|_{q_1^{**},Q} + \|\mathbf{A}\|_{p,Q}\Big) \le C\Big(1 + \|\mathbf{f}\|_{q_1,Q} + \|\mathbf{A}\|_{p,Q}\Big).$$

Hence

$$\|D\mathbf{u}\|_{r_1,Q} \le C(1 + \|\mathbf{f}\|_{q_1,Q} + \|\mathbf{A}\|_{p,Q}).$$

Applying the bootstrapping arguments , we obtain as in the elliptic case that the theorem holds after k iterations with

$$k \ge \min\left\{ \Big[\frac{\log(r_0/(r_0-2))}{\log((n+4)/(n+2))}\Big], \Big[\log\frac{r_0[q(n+4)-2(n+2)]}{q(r_0-2)(n+4)}\Big/\log\frac{n+4}{n+2}\Big] + 1 \right\}.$$

$\square$

Acknowledgments. The author is indebted to the referee for the valuable remarks which have lead to improvement of the article.

References

[1] Acerbi, E., Mingione, G., Gradient estimates for a class of parabolic systems, *Duke Math. J.* **136** (2007), 285-320.

[2] Arkhipova, A.A., Reverse Hölder inequalities with boundary integrals and L_p-estimates for solutions of nonlinear elliptic and parabolic boundary-value problems, *Amer. Math. Soc. Transl.* **164** (1995), 15–42.

[3] Arkhipova, A.A., L_p-estimates of the gradients of solutions of initial/boundary-value problems for quasilinear parabolic systems, *J. Math. Sci.* **73** (1995), 609–617.

[4] Arkhipova, A.A., Reverse Hölder inequalities in parabolic problems with anisotropic space data, Some Applications of Functional Analysis to Problems of Mathematical Physics (S.K. Godunov, editor), Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1992, 3–22 (in Russian).

[5] Arkhipova, A.A., Ladyzhenskaya, O.A., On a modification of Gehring's lemma, *J. Math. Sci.* (5) **109** (2002), 1805–1813.

[6] Byun, S.-S., Optimal $W^{1,p}$ regularity theory for parabolic equations in divergence form, *J. Evol. Equ.* (3) **7** (2007), 415–428.

[7] Byun, S.-S., Ryu, S., Global estimates in Orlicz spaces for the gradient of solutions to parabolic systems, *Proc. Amer. Math. Soc.* (2) **138** (2010), 641–653.

[8] Byun, S.-S., Wang, L.,Gradient estimates for elliptic systems in non-smooth domains, *Math. Ann.* **341** (2008), 629–650.

[9] Campanato, S., *Sistemi ellittici in forma divergenza. Regolarità all'interno,* Pubblicazioni della Classe di Scienze: Quaderni, Scuola Norm. Sup., Pisa, 1980.

[10] De Giorgi, E., Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolary, Mem. Accad. sci. Torino Cl. Sci. Fis. Mat. Natur. (3) **3** (1957), 25–43.

[11] Dong, H., Kim, D., Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth, *Comm. Part. Differ. Equ.* **36** (2011), 1750-1777.

[12] Duzaar, F., Grotowski, J.F., Optimal regularity for nonlinear elliptic systems: the method of A-harmonic approximation, *Manuscripta Math.,* **103** (2000), 267–298.

[13] Duzaar, F., Kristensen, J., Mingione, G., The existence of regular boundary points for nonlinear elliptic systems, *J. Reine Angew. Math.,* **602** (2007), 17–58.

[14] Duzaar, F., Mingione, G., Steffen, K., *Parabolic systems with polynomial growth and regularity,* Mem. Am. Math. Soc. **1005**, 2011.

[15] Feng, Z., Zheng, S., Regularity for quasi-linear elliptic systems with discontinuous coefficients, *Dyn. Partial Differ. Eq.* **5** (2008), 87–99.

[16] Gehring, F.W., L_p-integrability of the partial derivatives of a quasi conformal mapping, *Acta Math.* **130** (1973), 265–277.

[17] Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems,* Annals of Mathematics Studies, 105, Princeton University Press, Princeton, NJ, 1983.

[18] Giaquinta, M., Modica, G., Regularity results for some classes of higher order nonlinear elliptic systems, *J. Reine Angew. Math.* **311/312** (1979), 437–451.

[19] Giaquinta, M., Struwe, M., On the partial regularity of weak solutions of nonlinear parabolic systems, *Math. Z.* **142** (1975), 67–86.

[20] John, F., Nirenberg, L., On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **14** (1961), 415–426.

[21] John, O., Stara, Y., On the regularity and nonregularity of elliptic and parabolic systems, Equadiff-7, Proc. Conf. Prague, 1989; J. Kurzweil ed., Teubner-Texte Math. **118**, Teubner, Leipzig, 1990, 28–36.

[22] Krylov, N. V., Parabolic and elliptic equations with VMO coefficients, *Commun. Partial Differ. Equ.* (3) **32** (2007), 453-475.

[23] Krylov, N. V., *Lectures on Elliptic and Parabolic Equations in Sobolev spaces*, Graduate Studies in Mathematics 96, Amer. Math. Soc., Providence, R.I. (2008).

[24] Ladyzhenskaya, O.A., Ural'tseva, N.N., *Linear and Quasilinear Equations of Elliptic Type,* 2nd Edition, Nauka, Moscow, 1973, (in Russian).

[25] Ladyzhenskaya, O.A., Solonnikov, V.A., Ural'tseva, N.N., *Linear and Quasilinear Equations of Parabolic Type,* Transl. Math. Monographs 23, Amer. Math. Soc., Providence, R.I. (1968).

[26] Leonetti, F., Petricca, P.V., Regularity for solutions to some nonlinear elliptic systems, *Complex Var. Elliptic Eq.* (12) **56** (2011), 1099-1113.

[27] Marino, M., Maugeri, A., L^p-theory and partial Hölder continuity for quasilinear parabolic systems of higher order with strictly controlled growth, *Ann. Mat. Pura Appl.* **139** (1985), 107–145.

[28] Maugeri, A., Palagachev, D.K., Softova, L.G., *Elliptic and Parabolic Equations with Discontinuous Coefficients,* Math. Res., 109, Wiley-VCH, Berlin, 2000.

[29] Milakis, E., Toro, T., Divergence form operators in Reifenberg flat domains, *Math. Z.* **264** (2010), 15–41.

[30] Nash, J., Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.* **80** (1958), 931–954.

[31] Palagachev, D.K., Discontinuous superlinear elliptic equations of divergence form, *NoDEA Nonlinear Differential Equations Appl.* **16** (2009), 811–822.
[32] Palagachev, D.K., Global Hölder continuity of weak solutions to quasilinear divergence form elliptic equations, *J. Math. Anal. Appl.* **359** (2009), 159–167.
[33] Palagachev, D.K., Quasilinear divergence form elliptic equations in rough domains, *Complex Var. Elliptic Eq.* **55** (2010), 581–591.
[34] Palagachev, D.K., Softova, L.G., Characterization of the interior regularity for parabolic systems with discontinuous coefficients, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **16** (2005), 125–132.
[35] Palagachev, D.K., Softova, L.G., Fine regularity for elliptic systems with discontinuous ingredients, *Arch. Math.* **86** (2006), 145–153.
[36] Palagachev, D.K., Softova, L.G., Divergence form parabolic equations in Reifenberg flat domains, *Discr. Cont. Dynam. Syst.* (4) **31** (2011), 1397–1410.
[37] Palagachev, D.K., Softova, L.G., The Calderón-Zygmund property for quasilinear divergence form equations over Reifenberg flat domains, *Nonlinear Anal.* **74** (2011), 1721–1730.
[38] Reifenberg, E.R., Solution of the Plateau problem for m-dimensional surfaces of varying topological type, *Acta Math.* **104** (1960), 1–92.
[39] Toro, T., Doubling and flatness: geometry of measures, *Notices Amer. Math. Soc.* **44** (1997), 1087–1094.

Department of Civil Engineering, Second University of Naples, Via Roma 29, 81031 Aversa, Italy
E-mail address: luba.softova@unina2.it